KASR TARTIBLI TO‘LQIN TENGLAMALARI UCHUN FAZOVIY O‘ZGARUVCHILAR BO‘YICHA NOLOKAL SHARTLI CHEGARAVIY MASALA HAQIDA

Alimov Z.
Urganch Davlat universiteti "Matematika (differensial tenglama va matematik fizika)" yo‘nalishi 2-bosqich magistranti

Olloyarov M.SH
Urganch Davlat universiteti "Matematika (differensial tenglama va matematik fizika)" yo‘nalishi 2-bosqich magistranti

https://doi.org/10.5281/zenodo.8018167

Аннотация: *В этой статье исследуется однозначное решение нелокальной условной задачи о пространственных переменных для волнового уравнения дробного порядка. Это был метод разделения переменных с использованием решения задачи Коши для уравнения дробного порядка, а также методов оценки функции миттага-Леффлера.*
Ключевые слова: *функция миттага-Леффлера, дробная упорядоченная производная Гильфера; волновое уравнение; задача Коши; метод Фурье.*
Annotation: *this article explores the one-valued solvability of a non-trivial conditional issue on spatial variables for a fractional order wave equation. This involved a method of separation of variables, using the Cauchy problem solution for the fractional order equation as well as Mittag-Leffler function estimation methods.*
Keywords: *Mittag-Leffler function, Hilfer fraction ordered derivative; wave equation; Cauchy question; Fourier method.*
Annotatsiya: *Ushbu maqolada kasr tartibli to‘lqin tenglamasi uchun fazoviy o‘zgaruvchilar bo‘yicha nolokal shartli masalaning bir qiymatli yechilishi tadqiq etilgan. Bunda o‘zgaruvchilarni ajratish usuli bo‘lib, kasr tartibli tenglama uchun Koshi masalasi yechimi hamda Mittag-Leffler funksiyasi baholash usullaridan foydalanilgan.*
Kalit so‘zlar: *Mittag-Leffler funksiyasi, Xilfer kasr tartibli hosilasi; to‘lqin tenglamasi; Koshi masalasi; Furye usuli.*

Ma’lumki, xususiy hosilali differensial tenglamalar fizika, mexanika, biologiya, kimyo va boshqa ko‘plab amaliyotga bog‘langan fanlardagi jarayonlarni matematik modellashtirishda muhim ahamiyat kasb etadi [1-4]. Oxirgi yillarda bunday matematik modellarni mukkamallashtirishda kasr tartibli hosilalar qatnashgan turli xususiy hosilali differensial tenglamalardan foydalanilyapti [5-6]. Bunga kasr tartibli analizning o‘ziga xos usullari ishlab chiqilgani ham asos bo‘lib xizmat qilmoqda [7-8]. Kasr tartibli hosilalar asosan vaqt o‘zgaruvchisiga nisbatan qo‘llanilayotganining ham o‘ziga xos sabablari bo‘lib, bu xotira effekti, g‘ovak muhitlar bilan bog‘liqdir [9]. Tadqiqotning asosiy usuli o‘zgaruvchilarni ajratish usuli bo‘lib, kasr tartibli tenglama uchun Koshi masalasi yechimi hamda Mittag-Leffler funksiyasi bahosidan foydalanamiz. Fazoviy o‘zgaruvchilar bo‘yicha nolokal shart berilganda Furye usuli (o‘zgaruvchilarni ajratish usuli) o‘z-o‘ziga qo‘shma bo‘lmagan spektral masala bilan ishlashni taqazo qiladi. Bu holatlarda yechim biortogonal qator ko‘rinishida qidiriladi. Bizning holatda nolokal shartlar bunday holatga kelmaydi. O‘z-o‘ziga qo‘shma masala tufayli sinus-Furye qatoriga yoyilgan yechimni tadqiq etishda katta qiyinchiliklarga duch kelmaymiz. Bunday nolokal shartlarning kelib chiqishi Ionkin-Samarskiylarning ishiga borib taqaladi va muayyan fizik jarayonni matematik modellashtirishda qo‘llaniladi.

Quyidagi

$$D_{0t}^{(\alpha,\beta)\mu}u(x,y,t) = U_{xx}(x,y,t) + U_{yy}(x,y,t) \tag{1}$$

tenglamani $\Omega = \{(x,y,t): 0 < t < T, 0 < x < 1, 0 < y < 1\}$ sohada tadqiq etamiz. Bu yerda α, β -shunday haqiyqiy sonlarki, $1 < \alpha, \beta \le 2$ va

$$D_{ot}^{(\alpha,\beta)}f(t) = I_{ot}^{\mu(2-\alpha)}\left(\frac{d}{dt}\right)^2 I_{ot}^{(1-\mu)(2-\beta)}f(t) \tag{2}$$

α, β tartibli , $\mu \in [0,1]$ tipdagi umumlashgan Xilfer kasr tartibli operator [10, 11], I_{ot}^{φ} esa

$$I_{ot}^{\varphi}f(t) = \frac{1}{\Gamma(\varphi)}\int_0^t (t-z)^{\varphi-1}f(z)dz,\ t > 0 \tag{3}$$

formula bilan aniqlanuvchi φ -tartibli Riman-Liuvill integral operatori [7].

(1) tenglama uchun Ω sohada quyidagi masalani taqdim etamiz:

Masala. (1) tenglamaning Ω sohadagi quyidagi shartlarni qanoatlantiruvchi regulyar yechimi topilsin:

$$u_x(0,y,t) - h_1u(0,y,t) = 0,\ u_x(1,y,t) - h_2u(1,y,t) = 0,\ 0 \le y \le 1, 0 < t \le T, \tag{4}$$

$$u_y(x,0,t) - k_1u(x,0,t) = 0,\ u_y(x,1,t) - k_2u(x,1,t) = 0,\ 0 \le x \le 1, 0 < t \le T, \tag{5}$$

$$\lim_{t\to 0} I_{0t}^{(1-\mu)(1-\beta)}u(x,y,t) = \varphi(x,y),\ 0 \le x \le 1,\ 0 \le y \le 1, \tag{6}$$

$$\lim_{t\to 0}\frac{d}{dt}I_{ot}^{(1-\mu)(2-\beta)}u(x,y,t) = \psi(x,y),\ 0 < x < 1,\ 0 < y < 1, \tag{7}$$

bu yerda $\varphi(x,y), \psi(x,y)$ - berilgan funksiyalar, h_i, k_i $(i=1,2)$ - berilgan haqiqiy sonlar.

Ta’rif. (1) tenglamaning Ω sohadagi **regulyar yechimi** deb shunday $u(x,y,t)$ funksiyaga aytiladiki u

$$D_{o_t}^{(\alpha,\beta)\mu}u(x,y,t) \in C(\Omega),\ u_{xx}(x,y,t),\ u_{yy}(x,y,t) \in C(\Omega),\ t^{(1-\mu)(2-\beta)}u(x,y,t) \in C(\overline{\Omega})$$

regulyarlik shartlarini hamda Ω sohada (1) tenglamani qanoatlantiradi.

Masalani o‘zgaruvchilarni ajratish usuli bilan tadqiq etamiz. Bunda masala yechimi

$$u(x,y,t) = \sum_{m,n=0}^{\infty} U_{m,n}(t)X_m(x)Y_n(y) \tag{8}$$

ko‘rinishda qidiriladi, bu yerda $U_{m,n}(t)$ noma’lum funksiyalar bo‘lib, ularni topish uchun keyinchalik mos nolokal shartli masalaga kelinadi. Bu yerda $X_m(x),\ Y_n(y)$ funksiyalar sistemasi Furye usulini qo‘llashdan kelib chiqqan quyidagi spectral masalaning xos funksiyalaridir:

$$Z''(z)+\mu^2 Z(z)=0,\ 0<z<1;$$
$$Z'(0)-d_1 Z(0)=0,\ Z'(1)+d_2 Z(1)=0.$$

Bu masalaning xos sonlari

$$ctg\,\mu=\frac{\mu}{d_1+d_2}-\frac{d_1 d_2}{\mu(d_1+d_2)}$$

tenglamaning musbat ildizlari, ularga mos xos funksiyalar esa

$$Z_k(z)=\sqrt{c_k}\left(\mu_k\cos\mu_k z+d_1\sin\mu_k z\right),\ k\in\mathbf{N}$$

$$c_k=\frac{2\left(\mu_k^2+d_2^2\right)}{\left(\mu_k^2+d_1^2\right)\left(\mu_k^2+d_2^2\right)+\left(d_1+d_2\right)\left(\mu_k^2+d_1 d_2\right)}.$$

Bu xos funksiyalar sistemasi kvadrati bilan jamlanuvchi funksiyalar sinfida to‘la ortonormal sistema tashkil etadi.

$U_{m,n}(t)$ noma’lum funksiyalarni topish uchun (8) ni (1) tenglamaga va (6), (7) boshlang‘ich shartlarga qo‘yamiz va t o‘zgaruvchiga nisbatan quyidagi boshlang‘ich masalani olamiz:

$$\begin{cases} D_{0t}^{(\alpha,\beta)\mu}U_{m,n}(t)+\lambda_{m,n}U_{m,n}(t)=0,\\ \lim\limits_{t\to 0} I_{0t}^{(1-\mu)(1-\beta)}U_{m,n}(t)=\varphi_{m,n},\\ \lim\limits_{t\to 0}\frac{d}{dt}I_{0t}^{(1-\mu)(1-\beta)}U_{m,n}(t)=\psi_{m,n}.\end{cases} \tag{9}$$

Bu yerda $\lambda_{m,n}=(n\pi)^2+(m\pi)^2,\ m,n\in\mathbf{N},\ \varphi_{m,n}\ va\ \psi_{m,n}$ lar esa mos ravishda $\varphi(x,y)$ va $\psi(x,y)$ funksiyalarning Furye koeffitsiyentlaridir.

(9) masalaning yechimi

$$U_{m,n}(t)=\varphi_{m,n}t^{\gamma-1}E_{\delta,\gamma}\left(-\lambda_{m,n}t^{\delta}\right)+\psi_{m,n}t^{\gamma-2}E_{\delta,\gamma-1}\left(-\lambda_{m,n}t^{\delta}\right) \tag{10}$$

ko‘rinishida yoziladi. Bu yerda

$$\gamma=\beta+\mu(2-\beta),\ \delta=\beta+\mu(\alpha-\beta),\ E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}\ -$$

ikki karrali Mittag-Leffler funksiyasi [4].

(8) formula bilan berilgan cheksiz qatorning tekis yaqinlashishini isbotlash uchun $U_{m,n}(t)$ ni baholab olamiz. Buning uchun

$$\left|E_{\alpha,\beta}(z)\right|\le\frac{C}{1+|z|} \tag{11}$$

bahodan foydalanamiz [4].

$$\left|U_{m,n}(t)\right| \le \left|\varphi_{m,n}\right| \times \frac{\left|t^{\gamma-1}\right| \times C}{1+\left|\mu_{m,n} t^{\delta}\right|} + \left|\psi_{m,n}\right| \times \frac{c\left|t^{\gamma-2}\right|}{1+\left|\mu_{m,n} t^{\delta}\right|} + \left|\varphi_{m,n}\right| \times \frac{\left|x^{\gamma-1}\right| \times C}{1+\left|\mu_{m,n} x^{\delta}\right|} + \left|\psi_{m,n}\right| \times \frac{c\left|x^{\gamma-2}\right|}{1+\left|\mu_{m,n} x^{\delta}\right|} \quad (12)$$

yoki

$$\left|U_{m,n}(t)\right| \le \frac{c_1}{\left|\mu_{m,n}\right|} \times \frac{\left|\varphi_{m,n}\right|}{\left|t^{\delta+1-\gamma}\right|} + \frac{c_2}{\left|\mu_{m,n}\right|} \times \frac{\left|\psi_{m,n}\right|}{\left|t^{\delta+2-\gamma}\right|}. \quad (13)$$

(12) ni (13) shaklda yozib olish bizni ahamiyati kata bo‘lmagan konstantalarni ixchamroq yozib olishga imkon beradi. (13) ni hisobga olsak (8) dan

$$\left|u(x,y,t)\right| \le \sum_{m,n=0}^{\infty} \frac{\left|\varphi_{m,n}\right|}{\left|\mu_{m,n}\right|} \frac{c_1}{\left|t^{\delta+1-\gamma}\right|} + \frac{\left|\psi_{m,n}\right|}{\left|\mu_{m,n}\right|} \frac{c_2}{\left|t^{\delta+2-\gamma}\right|} \quad (14)$$

ni olish mumkin. $u_{xx}(x,y,t)$, $u_{yy}(x,y,t)$ funksiyalarga mos keluvchi cheksiz qatorlar uchun

$$\left|u_{xx}(x,y,t)\right| \le \sum_{m,n=0}^{\infty} \left(\frac{c_3}{\left|t^{\delta+1-\gamma}\right|}\left|\varphi_{m,n}\right| + \frac{c_4}{\left|t^{\delta+2-\gamma}\right|}\left|\psi_{m,n}\right| \right),$$

$$\left|u_{yy}(x,y,t)\right| \le \sum_{m,n=0}^{\infty} \left(\frac{c_3}{\left|t^{\delta+1-\gamma}\right|}\left|\varphi_{m,n}\right| + \frac{c_4}{\left|t^{\delta+2-\gamma}\right|}\left|\psi_{m,n}\right| \right) \quad (15)$$

baho kerak bo‘ladi. (14) va (15) da $\left|\varphi_{m,n}\right| = \frac{\left|\varphi_{m,n}^{(1)}\right|}{\mu_{m,n}}, \left|\psi_{m,n}\right| = \frac{\left|\psi_{m,n}^{(1)}\right|}{\mu_{m,n}}$ va Parseval tengligidan foydalansak

$$\left|u(x,y,t)\right| \le c_5 + \left\|\varphi\right\|_2^2 + \left\|\psi\right\|_2^2, \quad \left|u_{xx}(x,y,t)\right| \le c_6 + \left\|\varphi_x'\right\|_2^2 + \left\|\psi_x'\right\|_2^2,$$

$$\left|u_{yy}(x,y,t)\right| \le c_7 + \left\|\varphi_y'\right\|_2^2 + \left\|\psi_y'\right\|_2^2$$

ni olamiz. Bu esa $u(x,y,t)$ *va* $u_{xx}(x,y,t)$ funksiyalarga mos keluvchi cheksiz qatorlarning tekis yaqinlashishini isbotlaydi.

Masala yechimining yagonaligi fazoviy o‘zgaruvchilar bo‘yicha olingan sistemalarning to‘la ortonormal basis tashkil etishidan kelib chiqadi [12, 13]. Demak, quyidagi tasdiq o‘rinli:

Teorema. *Agar*

$$\varphi(x,y), \psi(x,y) \in C\left([0,1]\times[0,1]\right),$$

$$\varphi_x(x,y), \psi_x(x,y) \in L_2\left((0,1)\times(0,1)\right), \varphi_y(x,y), \psi_y(x,y) \in L_2\left((0,1)\times(0,1)\right)$$

bo‘lsa, u holda masala yagona yechimga ega va u (8) ko‘rinishda topiladi.

Foydalanilgan adabiyotlar ro‘yxati

1. Courant, R., Hilbert, D. (1962). Methods of Mathematical Physics, vol. II, New York: Wiley-Interscience.
2. Gershenfeld, N. (1999). The Nature of Mathematical Modeling (1st ed.), New York: Cambridge University Press.

3. Wazwaz, Abdul-Majid (2009). Partial Differential Equations and Solitary Waves Theory. Higher Education Press.
4. Roubíček, T. (2013). Nonlinear Partial Differential Equations with Applications. International Series of Numerical Mathematics, vol. 153 (2nd ed.), Basel, Boston, Berlin: Birkhäuser.
5. Uchaikin V.V. (2013). Fractional Derivatives for Physicists and Engineers. Vol.1. Background and Theory. Springer, Berlin, Heidelberg.
6. Uchaikin V.V. (2013). Fractional Derivatives for Physicists and Engineers. Vol.2. Applications. Springer, Berlin, Heidelberg.
7. Kilbas A.A., Srivastava H.M., Trujillo J.J. (2006). Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam.
8. Podlubny I. (1999). Fractional Differential Equations. Academic Press, San Diego.
9. Nigmatullin R.R. (1984). On the theory of relaxation with “remnant” memory. Phys. Status Solidi B, 124(1), pp.389-393.
10. De Oliveira, E.C.; Tenreiro Machado, J.A. (2014). A review of definitions for fractional derivatives and integral. Math. Probl. Eng., 238459.
11. Bulavatsky V.M. (2014). Closed form of the solutions of some boundary problems for anomalous diffusion equation with Hilfer’s generalized derivative. Cybernetics and Systems Analysis. 30(4), pp. 570-577.
12. Alimov Z. S. (2022). On a boundary-value problem for the degenerate wave equation in a prismatic domain. Bull. Inst. Math., Vol. 5, No3, pp. 36-40.
13. David J., Nolte A., Sherman J. (2018). A Boundary-Value Problem for 3-D Fractional Wave Equation with Singularity. Bull. Inst. Math., Vol. 1, No2, pp. 28-52.